Discussion 18 Worksheet

Parametric surfaces and surface integrals

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MATH 53 Multivariable Calculus

1 Sphere Parametrization

Consider a sphere of radius R centered at the origin. We know that the sphere can be parametrized by

$$\vec{r}(\phi, \theta) = \begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix},$$

$0 \le \phi \le \pi, 0 \le \theta \le 2\pi$.

(a) Compute the partial derivatives of $\vec{r}(\phi, \theta)$.

(b) Compute the normal vector $\vec{r}_u \times \vec{r}_v$ produced by this parametrization. Express it in terms of ϕ, θ and x, y, z.

(c) Use the magnitude of the normal vector (the "Jacobian") to compute the area of the unit sphere.

(d) Compute the surface integral of z^2 over the sphere.

2 Surface Areas

Parametrize the following surfaces in an appropriate way (if they are not already parametrized) and compute their normal vectors and area.

(a) The portion of the elliptic paraboloid $z = x^2 + y^2$ lying over the unit disk.

(b) The ellipsoid $2z^2 + x^2 + y^2 = 1$. You don't need to evaluate the integral, but you can do it using

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right) + C.$$

(c) The parametric surface $\vec{r}(u, v) = (u^2, uv, v^2/2)$ where $0 \le u \le 1, 0 \le v \le 2$.

(d) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

3 Scalar Surface Integrals

Compute the surface integral

$$\iint_S f(x, y, z)\,dS$$

for the given function $f(x, y, z)$ over the surface S.

(a) $f(x, y, z) = x$ where S is the surface $y = x^2 + 4z, 0 \le x \le 1, 0 \le z \le 1$.

(b) $f(x, y, z) = (x^2 + y^2)z$ and S the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.